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Squares of S-functions of special shapes

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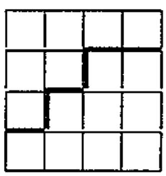
Abstract. Three specific problems are introduced and solved. The first is to determine the number of times the adjoint representation of SU_n occurs in the Kronecker square of a self-contragredient representation. The second is to determine the number of times the adjoint appears in the symmetric part of the square, with the third being the number of times the adjoint appears in the antisymmetric part. These problems are solved by recasting them as three problems concerning the squares of self-complementary S-functions and an equivalent adjoint S-function of a particular shape.

1. Introduction

Problems in physics and mathematics can often be expressed in terms of ordered partitions of integers associated with special shapes and special types of Schur functions (S-functions). In this paper we study a problem that arose in the study of properties of the real representations of the groups SU_n. The entire problem may be recast as a particular problem in the theory of S-functions. We shall first introduce some definitions and then give a precise statement of the S-function problem followed by its solution. Finally, we relate the results to irreducible representations (irreps) of the special unitary groups SU_n. Throughout this paper we follow the notation described in the book by Macdonald (Macdonald 1979).

2. Some definitions

Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell)$ be an ordered partition such that $(\lambda_1 \geqslant \lambda_2 \geqslant \ldots \geqslant \lambda_\ell \geqslant 0)$, possibly with trailing zeros to make the total number of parts of λ equal to a positive integer n. Such a partition may be inscribed in a box $B = (\lambda_1^n)$ having λ_1 columns and n rows as illustrated for the particular case of the partition (4210).



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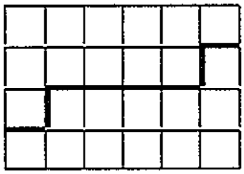
The cells in the lower portion of B not occupied by those of λ describe the shape of a partition λ^c (after rotation by π) which we shall term the *complement* of λ where

$$\lambda^c = (\lambda_1 - \lambda_n, \lambda_1 - \lambda_{n-1}, \ldots, 0) \tag{1}$$

A partition λ^{sc} will be said to *self-complementary* if $\lambda = \lambda^c$. In that case the box B involves two equal parts (to within a rotation by π) as shown below for the partition (6510).



We may associate with any self-complementary partition λ^{sc} an *adjoint equivalent* partition λ^{ae} such that

$$\lambda^{ae} = (\lambda_1 + 1, \lambda_1^{n-2}, \lambda_1 - 1) \tag{2}$$

of length $\ell(\lambda^{ae}) = n$ and weight $|\lambda^{ae}| = 2|\lambda^{sc}|$. Thus $\lambda^{sc} = (6510)$ is associated with the adjoint equivalent partition $\lambda^{ae} = (7665)$.

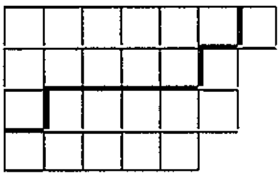
The shape of λ^{ae} for a given λ^{sc} is obtained by the simple procedure

(i) draw the $\lambda_1 \times n$ box B;

(ii) add one cell to the first row of B; and

(iii) delete the south-eastern corner cell of B.

The shape for $\lambda^{ae} = (7665)$ is illustrated below.



3. Statement of the problem

Let the S-function indexed by the partition λ be denoted by $\{\lambda\}$. Denote the coefficient of $\{\lambda\}$ in the expansion of a symmetric function F by $\langle F, \{\lambda\}\rangle$. The three problems to be considered involve the evaluation of the following non-negative integers

$$\langle\{\lambda^{sc}\} \cdot \{\lambda^{sc}\}, \{\lambda^{ae}\}\rangle \tag{I}$$
$$\langle\{2\} \circ \{\lambda^{sc}\}, \{\lambda^{ae}\}\rangle \tag{II}$$
$$\langle\{1^2\} \circ \{\lambda^{sc}\}, \{\lambda^{ae}\}\rangle. \tag{III}$$

4. Some propositions

The explicit evaluation of (I) can be made as follows.

It is known that $\langle\{\lambda\}\cdot\{\mu\},\{\gamma\}\rangle = \langle\{\lambda\},\{\gamma\}/\{\mu\}\rangle$.

To find the coefficient, we:

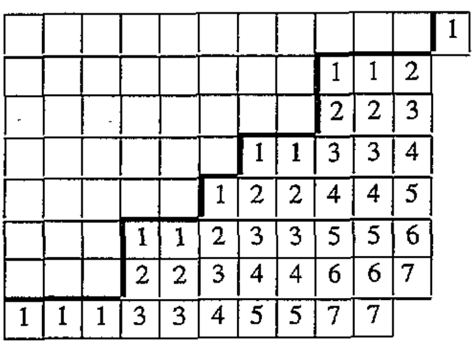
(a) draw the shape $\lambda^{ae}/\lambda^{sc}$;

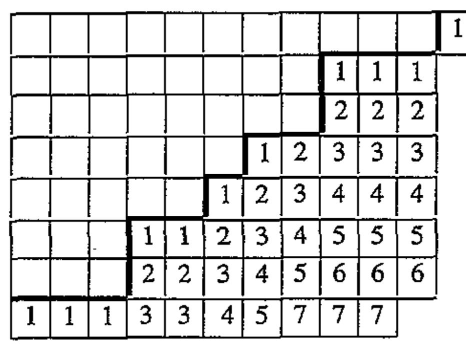
(b) fill the shape with λ_i^{sc} copies of i for $1 \leqslant i \leqslant \ell(\lambda^{sc})$, such that the numbers must be weakly increasing from left to right for any row and strictly increasing from top to bottom for any column; and

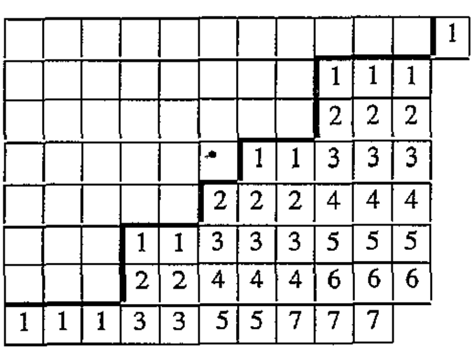
(c) further, if the letters are read from right to left, row by row, then the word must be a lattice permutation.

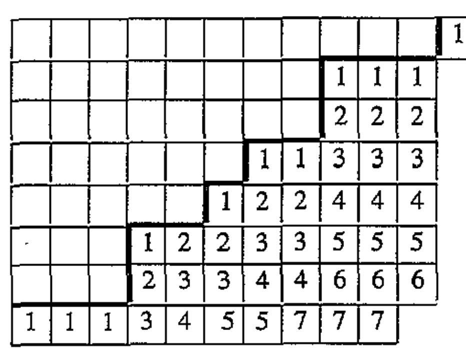
Thus for $\lambda^{sc} = (11\ 8865330)$ we obtain the five numberings below to give

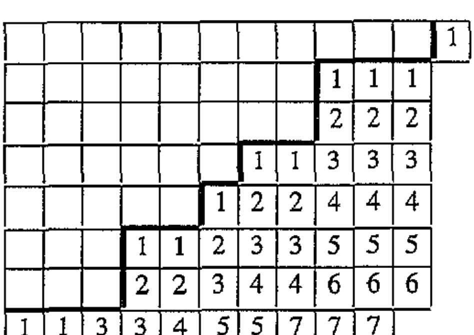
$$\langle\{11\ 8865330\}\cdot\{11\ 8865330\},\{12\ 11^6\ 10\}\rangle = 5.$$











Examples such as the above suggested the following proposition:

Proposition 1.

$$\langle\{\lambda^{sc}\}\cdot\{\lambda^{sc}\},\{\lambda^{ae}\}\rangle = d(\lambda^{sc}) \tag{3}$$

where $d(\lambda^{sc})$ is the number of distinct non-zero parts of the partition λ^{sc}.

The above proposition comes from first noting that

$$\{\lambda/1\}\cdot\{1\} \supseteq d(\lambda)\{\lambda\}$$

(with all other terms having multiplicities $\leqslant 1$.) and then that the shape of $\{\lambda^{ae}/\lambda^{sc}\}$ decomposes into two disjoint shapes, one containing just a single cell and the other a skew frame that has the same S-function content as $\{\lambda^{sc}/1\}$ leading to

$$\{\lambda^{ae}/\lambda^{sc}\} \equiv \{\lambda^{sc}/1\}\cdot\{1\}$$

and thence to the desired result of equation (3).

The square of an S-function $\{\lambda\}$ may be resolved into its symmetric and antisymmetric terms by use of the power sum plethysms

$$\{2\}\circ\{\lambda\} = \tfrac{1}{2}\left(p_1^2\circ\{\lambda\} + p_2\circ\{\lambda\}\right) \tag{4}$$

$$\{1^2\}\circ\{\lambda\} = \tfrac{1}{2}\left(p_1^2\circ\{\lambda\} - p_2\circ\{\lambda\}\right). \tag{5}$$

If $\langle p_2\circ\{\lambda^{sc}\},\{\lambda^{ae}\}\rangle = 0$ then

$$\langle\{2\}\circ\{\lambda^{sc}\},\{\lambda^{ae}\}\rangle = \langle\{1^2\}\circ\{\lambda^{sc}\},\{\lambda^{ae}\}\rangle = \tfrac{1}{2}\langle\{\lambda\}^2,\{\lambda^{ae}\}\rangle. \tag{6}$$

We may be assured that $\langle p_2\circ\{\lambda^{sc}\},\{\lambda^{ae}\}\rangle = 0$ if λ^{ae} has a non-null two-core (Littlewood 1951).

Proposition 2. If n is odd then λ^{ae} has a non-null two-core whereas if n is even then λ^{ae} has a null two-core.

Proof. The two-core of λ^{ae} is determined by the following steps.

(i) Make $\ell = \ell(\lambda^{ae})$ even (adding a zero if needed).

(ii) Form the augmented partition, $\hat{\lambda}^{ae}$, by adding the staircase $\Delta_\ell = (\ell-1,\ell-2,\ldots,3,2,1,0)$ to λ^{ae}, i.e. $\hat{\lambda}^{ae} = \lambda^{ae} + \Delta_\ell$.

(iii) Count the number of *even* and *odd* parts of $\hat{\lambda}^{ae}$. If the numbers are equal then λ^{ae} has a null two-core otherwise the two-core is non-null.

When n is *odd* we find that $\hat{\lambda}^{ae}$ has more even than odd parts and hence the two-core cannot be null, conversely for n *even* the number of odd and even parts is always equal and hence the two-core is always null. □

In the case when $\{\lambda^{ae}\}$ has null two-core, the coefficient $\langle p_2\circ\{\lambda^{sc}\},\{\lambda^{ae}\}\rangle$ is determined by the two-content of $\{\lambda^{ae}\}$ (Littlewood 1951). Let c_1, c_2 be the two-content of λ^{ae} and ε be the two-sign of λ^{ae}. Then $\{\lambda^{ae}\}$ appears in $p_2\circ\{\lambda^{sc}\}$ with coefficient k and sign ε if

$$\langle\{c_1\}\cdot\{c_2\},\{\lambda^{sc}\}\rangle = k. \tag{7}$$

The two-content of λ^{ae} is determined by:

Proposition 3. Let $\lambda^{ae} = (\lambda_1 + 1, \lambda_1^{n-2}, \lambda_1 - 1)$ with n even. If λ_1 is even then the two-content of λ^{ae} are

$$c_1 = \binom{\frac{\lambda_1+2}{2}}{}^{n/2} \qquad \text{and} \qquad c_2 = \left(\frac{\lambda_1-2}{2}\right)^{n/2} \tag{8}$$

while if λ_1 is odd then the two-content of λ^{ae} are

$$c_1 = \left(\frac{\lambda_1+1}{2}\right)^{n/2} \qquad \text{and} \qquad c_2 = \left(\frac{\lambda_1-1}{2}\right)^{n/2} \tag{9}$$

Proof. The two-content of λ^{ae} is obtained by reading the augmented partition $\hat{\lambda}^{ae}$ from right to left and replacing the first even number by 0, the second by 2, ... the $\frac{n}{2}$ even number by $n-2$. Similarly, the first odd number is replaced by 1, the second by 3, ..., the last odd number by $n-1$. Denote the resulting sequence by ρ^{ae}. Now, let

$$\hat{\lambda}^{ae} = \hat{\lambda}^{ae}_{\text{even}} \cup \hat{\lambda}^{ae}_{\text{odd}} \tag{10}$$

$$\hat{\rho}^{ae} = \hat{\rho}^{ae}_{\text{even}} \cup \hat{\rho}^{ae}_{\text{odd}} \tag{11}$$

where λ_{even} and λ_{odd} denote the partitions made from the even and odd parts of λ, respectively. Then the two-contents of λ^{ae} are

$$\tfrac{1}{2}(\hat{\lambda}^{ae}_{even} - \hat{\rho}^{ae}_{even}) \qquad \text{and} \qquad \tfrac{1}{2}(\hat{\lambda}^{ae}_{odd} - \hat{\rho}^{ae}_{odd}) \tag{12}$$

leading directly to the desired result. □

Proposition 4. If λ_1 is even then the two-sign of λ^{ae} is $\varepsilon = -1$.

If λ_1 is odd then the two-sign of λ^{ae} is $\varepsilon = -1$ if $n = 0 \bmod 4$ or if $n = 2 \bmod 4$ then $\varepsilon = +1$.

Proof. By definition the two-sign is equal to $(-1)^{\text{inv}(\rho^{ae})}$ where ρ^{ae} is as defined in proposition 3, and $\text{inv}(n_1, n_2, \ldots, n_\ell) = \sum_{i=1}^{\ell} \text{inv}(n_i)$, and $\text{inv}(n_i) = \text{card}\{n_j : j > i$ and $n_j > n_i\}$. This leads immediately to the proposition. □

Proposition 5. If n is even, $\lambda^{sc} \supset \{c_1\}$ and $\lambda^{sc} \supset \{c_2\}$, then,

$$\langle \{c_1\} \cdot \{c_2\}, \{\lambda^{sc}\} \rangle = 1. \tag{13}$$

Otherwise, the coefficient is zero.

In such a case all the S-functions of length $\leqslant n-1$ arising in $\{c_1\} \cdot \{c_2\}$ are indexed by self-complementary partitions. The above proposition is known (Stanley 1971).

5. Final results

With the above propositions established it becomes a simple matter to solve Problems (II) and (III) leading immediately to the following.

(i) If n is odd then

$$\langle\{2\}\circ\{\lambda^{sc}\},\{\lambda^{ae}\}\rangle = \langle\{1^2\}\circ\{\lambda^{sc}\},\{\lambda^{ae}\}\rangle = \tfrac{1}{2}d(\lambda^{sc}). \tag{14}$$

(ii) If n is even and $\lambda^{sc}\not\supset c_1$ and c_2 then

$$\langle\{2\}\circ\{\lambda^{sc}\},\{\lambda^{ae}\}\rangle = \langle\{1^2\}\circ\{\lambda^{sc}\},\{\lambda^{ae}\}\rangle = \tfrac{1}{2}d(\lambda^{sc}). \tag{15}$$

(iii) Suppose n is even, $\lambda^{sc}\supset c_1$ and $\lambda^{sc}\supset c_2$. If λ_1 is odd and $n=2 \bmod 4$, then

$$\langle\{2\}\circ\{\lambda^{sc}\},\{\lambda^{ae}\}\rangle = \frac{d(\lambda^{sc})+1}{2} \tag{16}$$

$$\langle\{1^2\}\circ\{\lambda^{sc}\},\{\lambda^{ae}\}\rangle = \frac{d(\lambda^{sc})-1}{2}. \tag{17}$$

(iv) Suppose n is even, $\lambda^{sc}\supset c_1$ and $\lambda^{sc}\supset c_2$. If λ_1 is even, or λ_1 is odd and $n=0$ mod 4, then

$$\langle\{2\}\circ\{\lambda^{sc}\},\{\lambda^{ae}\}\rangle = \frac{d(\lambda^{sc})-1}{2} \tag{18}$$

$$\langle\{1^2\}\circ\{\lambda^{sc}\},\{\lambda^{ae}\}\rangle = \frac{d(\lambda^{sc})+1}{2}. \tag{19}$$

We note that since the plethysm coefficients are integers, this proposition implies that when n is odd, or, when n is even but $\lambda^{sc}\not\supset c_1$ and c_2, then $d(\lambda^{sc})$ must be an even number. On the other hand, when n is even and $\lambda^{sc}\supset c_1$ and c_2, then $d(\lambda^{sc})$ must be an odd number. This result can also be verified directly by examining the shapes of λ^{sc}, c_1 and c_2.

6. Self-contragredient and adjoint representations in SU_n

The above problem was motivated by an analogous problem concerning representations of the Lie groups of the generic type SU_n. Let us recall a few well known properties of the irreducible representations of SU_n. The inequivalent irreducible representations of SU_n may be labelled by ordered partitions of integers involving at most $n-1$ non-zero parts. Those involving n parts are equivalent to irreps involving $\leqslant (n-1)$ parts via

$$\{\lambda_1,\lambda_2,\ldots,\lambda_n\}\equiv\{\lambda_1-\lambda_n,\lambda_2-\lambda_n,\ldots,0\} \tag{20}$$

i.e. one can remove any number columns of length n. The generators of the group SU_n span the real adjoint representation $\{21^{n-2}\}$.

The Kronecker product $\{\lambda\}\times\{\mu\}$ is equivalent to the Littlewood–Richardson evaluation of the S-function product $\{\lambda\cdot\mu\}$ with partitions involving more than n-parts are discarded and those with n-parts reduced to fewer than n-parts using (19).

An irrep of SU_n, $\{\lambda_1,\lambda_2,\ldots\}$, has a contragredient partner $\{\lambda_1-\lambda_n,\lambda_1-\lambda_{n-1},\ldots\}$. If

$$\{\lambda_1,\lambda_2,\ldots\}\equiv\{\lambda_1-\lambda_n,\lambda_1-\lambda_{n-1},\ldots\} \tag{21}$$

then the irrep $\{\lambda_1, \lambda_2, \ldots\}$ is said to be *self-contragredient* (SC). The irreps of SU_n are *real* only if they are self-contragredient. All other irreps of SU_n are *complex*. Thus the Kronecker square of an irrep $\{\lambda\}$ of SU_n will yield the adjoint irrep if and only if the irrep $\{\lambda\}$ is self-contragredient. Our problem was to determine the number of times the adjoint representation $\{21^{n-2}\}$ occurs in the Kronecker square of a given self-contragredient representation $\{\lambda^{sc}\}$ of SU_n and to determine the number of times it occurs in the symmetric and antisymmetric parts of the square. The relevant results follow directly from those found in the solution of the three S-function problems discussed earlier simply by noting the equivalence in (20) with, for given n

$$\{\lambda^{ae}\} \rightarrow \{21^{n-2}\}. \tag{22}$$

Acknowledgments

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References

Littlewood D E 1951 *Proc. R. Soc.* A **209** 333
Macdonald I G 1979 *Symmetric Functions and Hall Polynomials* (Oxford: Clarendon)
Stanley R P 1971 *Studies Appl. Math.* **50** 167–88, 259–79

† B G Wybourne, SCHUR is an interactive C package for calculating properties of Lie groups and symmetric functions distributed by: S Christensen, PO Box 16175, Chapel Hill, NC 27516 USA, e-mail: steve@scm.vnet.net. A detailed description can be seen by WEB users at http://scm.vnet.net/Christensen.html